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# A Bifurcation Phenomenon for the Periodic Solutions of the Duffing Equation

YUKIE KOMATSU
(小松 幸恵)

Department of Mathematics, Faculty of Science
Osaka University

## 1. INTRODUCTION AND RESULT

We announce here briefly the study developed in [3]. For details, we refer readers to that paper. We study a bifurcation phenomenon for the periodic solutions of the following Duffing equation which describes a nonlinear forced oscillation:

$$u''(t)+\mu u'(t)+\kappa u(t)+\alpha u^{3}(t)=f(t), \quad t\in\mathbb{R} \tag{1.1}$$

where $\mu$ and $\alpha$ are positive constants, $\kappa$ is a nonnegative constant, and $f(t)$ is a given periodic external force. It is known that for any periodic external force there exists at least one periodic solution of (1.1) with the same period as the external force. Furthermore, if the external force is suitably small, then the periodic solution is proved to be unique and asymptotically stable. On the other hand, in the case of the relatively large external force, numerical computations display a possibility of not only the non-uniqueness of the periodic solution but also the existence of various bifurcation phenomena. In particular, a strange attractor discovered by Ueda [7], so called Japanese attractor, is well known. However, it is surprising that there have been no mathematical proofs of a existence of bifurcation for the periodic solutions of (1.1). The aim of this paper is to give a mathematical proof of a existence of bifurcation for a special family of external force. To do that, we define the one-parameter families of periodic functions $\{u_\lambda(t)\}_{\lambda>0}$ and $\{f_\lambda(t)\}_{\lambda>0}$ with the period one by

$$\begin{cases} u_\lambda(t) := \lambda\sin 2\pi t\,, \qquad \lambda>0\,, \\ f_\lambda(t) := u''_\lambda(t)+\mu u'_\lambda(t)+\kappa u_\lambda(t)+\alpha u_\lambda^{3}(t), \end{cases} \tag{1.2}$$

so that the equation (1.1) has the trivial periodic solution $u(t)=u_\lambda(t)$ to the external force $f(t)=f_\lambda(t)$ for any $\lambda>0$. Then our main Theorem is

**Theorem 1.** *Suppose $\mu$ and $\kappa$ satisfy*

$$0\le\kappa<4\pi^2, \qquad \mu\le min\left(\frac{15(4\pi^2-\kappa)}{64\pi}\,,\,\frac{(16\pi^2-\kappa)^2}{384\pi^3}\right),$$

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*and the external force $f(t) = f_\lambda(t)$ is given by (1.2). Then there exist at least three positive constants $\Lambda_i$ $(i = 1,2,3;\quad \Lambda_1 < \Lambda_2 < \Lambda_3)$, which depend only on $\mu$ and $\kappa$ such that a nontrivial periodic solution of (1.1) with the period one bifurcates from $\{u_\lambda(t)\}_{\lambda>0}$ at $\lambda_i = \sqrt{\Lambda_i/\alpha}$ $(i = 1,2,3)$.*

To prove Theorem 1, we first reformulate the problem on the periodic solution of (1.1) to an integral equation in the section 2, and apply the Krasnosel'skii's Theorem [4] on bifurcation to the integral equation in the section 3. A crucial part in this process is to show the eigenvalue problem of the linealized equation at $u(t) = u_\lambda(t)$ has at least three simple eigenvalues. We investigate the eigenvalue problem in the section 4 by making use of the arguments on the continued fraction along the same line as in the paper Meshalkin and Sinai [5].

## 2. Reformulation of the problem

We shall seek the periodic solution of (1.1) with the period one in the form,

$$u(t) = u_\lambda(t) + \lambda v(t). \tag{2.1}$$

Substituting (2.1) to (1.1), we obtain the following problem :

$$\begin{cases} v''(t) + \mu v'(t) + \kappa v(t) + \alpha\lambda^2(L(v)(t) + N(v)(t)) = 0, \\ v(t+1) = v(t)\ , \qquad t \in \mathbb{R}. \end{cases} \tag{2.2}$$

where $L(v)$ and $N(v)$ are defined by

$$\begin{cases} L(v)(t) := 3v(t)\sin^2 2\pi t, \\ N(v)(t) := 3v^2(t)\sin 2\pi t + v^3(t). \end{cases} \tag{2.3}$$

We reformulate the problem (2.2) into an integral equation in the space $E$ defined by

$$E = \{u(t) \in C(\mathbb{R}) \quad ; \quad u(t+1) = u(t) \quad , \quad t \in \mathbb{R}\}. \tag{2.4}$$

It is noted that the space $E$ is Banach space, with the norm

$$\| u \| := \sup_{t\in[0,1]} |u(t)|.$$

We first consider the case $\kappa \neq 0$. It is easy to see that for any $f \in E$, the problem

$$\begin{cases} v''(t) + \mu v'(t) + \kappa v(t) = f(t), \\ v(t+1) = v(t) \quad , \quad t \in \mathbb{R}. \end{cases} \tag{2.5}$$

has a unique solution $v \in E \cap C^2(\mathbb{R})$. Let us denote this solution by $G(f)$. Then the problem (2.2) is reformulated to the following problem in $E$:

$$v = -\alpha\lambda^2 G(L(v) + N(v)). \tag{2.6}$$

Next in the case of $\kappa = 0$, we rewrite the problem (2.2) as

$$(2.7) \qquad \begin{cases} v'' + \mu v' = \alpha\lambda^2(L(v) + N(v)) - \alpha\lambda^2 \displaystyle\int_0^1 (L(v)(t) + N(v)(t))\, dt, \\ \displaystyle\int_0^1 v(t)\sin^2 2\pi t\, dt = -\frac{1}{3}\int_0^1 N(v)(t)\, dt, \\ v(t+1) = v(t), \quad t \in \mathbb{R}. \end{cases}$$

To solve (2.7), we consider the following two linear equations for any $f \in E$ and $\beta \in \mathbb{R}$,

$$(2.8) \qquad \begin{cases} v''(t) + \mu v'(t) = f(t) - \displaystyle\int_0^1 f(t)\, dt, \\ \displaystyle\int_0^1 v(t)\sin^2 2\pi t\, dt = 0, \quad v(t+1) = v(t), \quad t \in \mathbb{R}, \end{cases}$$

$$(2.9) \qquad \begin{cases} w''(t) + \mu w'(t) = 0, \\ \displaystyle\int_0^1 w(t)\sin^2 2\pi t\, dt = \beta. \end{cases}$$

It is standard to see that the problem (2.8) has a unique solution $v \in E \cap C^2(\mathbb{R})$, denoting it by $\widetilde{G}(f)$, and the solution of (2.9) is a just constant explicitly given by $2\beta$. Thus, the problem (2.2) with $\kappa = 0$ is reduced to the integral equation in $E$:

$$(2.10) \qquad v = -\alpha\lambda^2\widetilde{G}(L(v) + N(v)) - \frac{2}{3}\int_0^1 N(v)(t)\, dt.$$

## 3. Proof of Theorem 1

To show Theorem 1, we apply the Krasnosel'skii's Theorem [4] to the integral equation (2.6) (*resp* (2.10)) for $\kappa > 0$ (*resp* $\kappa = 0$).

**Theorem A (Krasnosel'skii's Theorem).** *Let $E$ be a Banach space and $f(x, \lambda)$ be a operator with domain $D \subset E \times \mathbb{R}$ into $E$ of the form,*

$$f(x, \lambda) = x - \lambda Tx + g(x, \lambda).$$

*Suppose the followings :*

(1) $\lambda_0 \neq 0, \quad (0, \lambda_0) \in D$.
(2) *$T$ is a linear compact operator in $E$.*
(3) *$g(x, \lambda)$ is a nonlinear compact operator of $D$ into $E$, which satisfies*
$g(0, \lambda) \equiv 0, \qquad g(x, \lambda) = o(\|x\|)$ *uniformly in the neighborhood* $\lambda = \lambda_0$.
(4) *$1/\lambda_0$ is an eigenvalue of $T$ with odd multiplicity.*

*Then $(0, \lambda_0)$ is a bifurcation point for $f(x, \lambda) = 0$.*

Now, let $E$ be a Banach space defined by (2.4) and $T$ be a operator in $E$ defined by

$$(3.1) \qquad Tv = \begin{cases} G(-L(v)) & if \quad \kappa \neq 0, \\ \widetilde{G}(-L(v)) & if \quad \kappa = 0, \end{cases}$$

for any $v \in E$, and $g(v,\lambda)$ be a operator with domain $D = E \times \mathbb{R}_+$ where $\mathbb{R}_+ = \{\lambda \in \mathbb{R};\quad \lambda > 0\}$, into $E$ defined by

$$(3.2)\qquad g(v,\lambda) = \begin{cases} G(\alpha\lambda^2 N(v)) & if \quad \kappa \neq 0, \\ \widetilde{G}(\alpha\lambda^2 N(v)) + \dfrac{2}{3}\displaystyle\int_0^1 N(v)(t)\,dt & if \quad \kappa = 0, \end{cases}$$

for any $v \in E$ and $\lambda \in \mathbb{R}_+$. Then the both integral equations (2.6) and (2.10) are equivalent to the equation :

$$(3.3)\qquad f(v,\lambda) := v - \alpha\lambda^2 Tv + g(v,\lambda) = 0.$$

Therefore, we may show the corresponding assumptions (1) $\sim$ (4) in Theorem A to the equation (3.3). These are verified by the following Propositions.

**Proposition 3.1.**

(i) $G(f)\,,\ \widetilde{G}(f) \in E \cap C^2(\mathbb{R})$ *for any* $f \in E$.

(ii) *There exist a positive constant* $C$ *such that for any* $f \in E$
$\| G(f) \|,\quad \| \widetilde{G}(f) \| \leq C\| f \|,$
$\| \frac{d}{dt}G(f) \|,\quad \| \frac{d}{dt}\widetilde{G}(f) \| \leq C\| f \|.$

(iii) $G$ *and* $\widetilde{G}$ *are compact operators in* $E$.

**Proposition 3.2.** *Suppose* $\mu$ *and* $\kappa$ *are positive constants satisfying the assumption of Theorem 1. Then there exist at least three positive constants* $\Lambda_i (i = 1,2,3,\quad \Lambda_1 < \Lambda_2 < \Lambda_3)$ *which depend only on* $\mu$ *and* $\kappa$ *such that* $\Lambda_i^{-1}$*are simple eigenvalues of* $T$ .

The proof of Proposition 3.1 is given by quite standard argument on ordinary differential equation, so omitted. We shall give the proof of Proposition 3.2 in the next section. Thus applying Theorem A to the equation (3.3), we can prove a nontrivial periodic solution of (3.3) bifurcates at $\lambda_i = \sqrt{\Lambda_i/\alpha}\,(i = 1,2,3)$. □

## 4. Eigenvalue problem of linearized equation

In this section, we give the proof of Proposition 3.2. First we note that the eigenvalue problem for $T$ is again equivalent to the problem:

$$(4.1)\qquad \begin{cases} w''(t) + \mu w'(t) + \kappa w(t) + 3\Lambda w(t)\sin^2 2\pi t = 0, \\ w(t+1) = w(t), \quad t \in \mathbb{R}, \end{cases}$$

where we set $\Lambda = \alpha\lambda^2$. We expand the solution by Fourier series as

$$(4.2)\qquad w(t) = \sum_{n=-\infty}^{\infty} a_n e^{2n\pi i t}, \qquad \{a_n\}_{n\in\mathbf{Z}} \in \ell^2.$$

Substituting (4.2) to (4.1), we obtain

$$\sum_{n=-\infty}^{\infty} (-4\pi^2 n^2 + 2\pi\mu n i + \kappa + 3\Lambda\sin^2 2\pi t) a_n e^{2n\pi i t} = 0,$$

which implies that $\{a_n\}_{n\in\mathbb{Z}}$ satisfies the following recurrence formula:

$$A_n(\Lambda)a_n + a_{n-2} + a_{n+2} = 0, \quad n \in \mathbb{Z}, \tag{4.3}$$

where

$$A_n(\Lambda) = -2 + \frac{16\pi^2 n^2 - 4\kappa}{3\Lambda} - \frac{8\pi\mu n i}{3\Lambda}.$$

We study this recurrence formula by sepalating the cases whether $n$ is odd or even. In the case $n = 2m+1$ ($m \in \mathbb{Z}$), setting $b_m = a_{2m+1}$ and $B_m(\Lambda) = A_{2m+1}(\Lambda)$, we rewrite (4.3) for $\{b_m\}_{m\in\mathbb{Z}}$ as

$$B_m(\Lambda)b_m + b_{m-1} + b_{m+1} = 0 \quad , \quad m \in \mathbb{Z}. \tag{4.4}$$

In the case $n = 2m$ ($m \in \mathbb{Z}$), setting $d_m = a_{2m}$ and $D_m(\Lambda) = A_{2m}(\Lambda)$, we rewrite for $\{d_m\}_{m\in\mathbb{Z}}$ as

$$D_m(\Lambda)d_m + d_{m-1} + d_{m+1} = 0 \quad , \quad m \in \mathbb{Z}. \tag{4.5}$$

For the solvability of these recurrence formulas (4.4) and (4.5), the following Lemma holds.

**Lemma 4.1.**
**(I)** *There exists $\Lambda_0 \in \mathbb{R}_+$ such that the nontrivial sequence $\{b_m(\Lambda_0)\}_{m\in\mathbb{Z}} \in \ell^2$ satisfies the recurrence formula (4.4), if and only if there exists $\Lambda_0 \in \mathbb{R}_+$ such that $\{B_m(\Lambda_0)\}_{m\in\mathbb{Z}}$ satisfies the condition,*

$$\mid B_0(\Lambda_0) - \mathfrak{B}(\Lambda_0) \mid = 1, \tag{4.6}$$

*where*

$$\mathfrak{B}(\Lambda) = \cfrac{1}{B_1(\Lambda) - \cfrac{1}{B_2(\Lambda) - \cfrac{1}{B_3(\Lambda) - \ddots}}}$$

**(II)** *There exists $\Lambda_0 \in \mathbb{R}_+$ such that the nontrivial sequence $\{d_m(\Lambda_0\}_{m\in\mathbb{Z}} \in \ell^2$ satisfies the recurrence formula (4.5), if and only if there exists $\Lambda_0 \in \mathbb{R}_+$ such that $\{D_m(\Lambda_0)\}_{m\in\mathbb{Z}}$ satisfies condition,*

$$D_0(\Lambda_0) = 2Re\mathfrak{D}(\Lambda_0), \tag{4.7}$$

*where*

$$\mathfrak{D}(\Lambda) = \cfrac{1}{D_1(\Lambda) - \cfrac{1}{D_2(\Lambda) - \cfrac{1}{D_3(\Lambda) - \ddots}}}$$

To prove Proposition 3.2, we may only show that there exist $\Lambda_i \in \mathbb{R}_+$ ( $i = 1,2,3$ ) which satisfy the equality (4.6) or (4.7) and correspond to the eigenvalues of $T$ with simple multiplicity. To do that, we make use of the following Worpitzky's Theorem [1] concerning the continued fractions.

**Theorem B (Worpitzky's Theorem).** *Let $\mathfrak{F}$ be a family of the formal continued fractions:*

$$\mathfrak{F} = \left\{ C = \cfrac{a_1}{1 + \cfrac{a_2}{1 + \cfrac{a_3}{1 + \ddots}}} \quad ; \quad a_k \in \mathbb{C}, \quad |a_k| \leq \frac{1}{4} \quad \text{for any } k \in \mathbb{N} \right\}$$

*Let $w_n(C)$ and $w(C)$ respectively denote the $n$-th approximant and the value of a convergent continued fraction $C$. Then a family $\mathfrak{F}$ is uniformly convergent, that is,*

$$\lim_{n \to \infty} \sup_{C \in \mathfrak{F}} |w_n(C) - w(C)| = 0.$$

*Furthermore, it holds that $|w(C)| \leq \frac{1}{2}$, for any $C \in \mathfrak{F}$.*

Now, let us define the constants $\{\widetilde{\Lambda}_i\}_{i=0}^5$ $(0 < \widetilde{\Lambda}_0 < \widetilde{\Lambda}_1 < \widetilde{\Lambda}_2 < \widetilde{\Lambda}_3 < \widetilde{\Lambda}_4 < \widetilde{\Lambda}_5)$ by

$$\widetilde{\Lambda}_0 = \frac{8(4\pi^2 - \kappa)}{21}, \qquad \widetilde{\Lambda}_1 = \frac{2(4\pi^2 - \kappa)}{3}, \qquad \widetilde{\Lambda}_2 = \frac{4(4\pi^2 - \kappa)}{3},$$

$$\widetilde{\Lambda}_3 = \frac{4(16\pi^2 - \kappa)}{9}, \qquad \widetilde{\Lambda}_4 = \frac{2(16\pi^2 - \kappa)}{3}, \qquad \widetilde{\Lambda}_5 = \frac{4(36\pi^2 - \kappa)}{9}.$$

According to Theorem B and the Intermediate Value Theorem, we can show that there exist constants $\{\Lambda_i\}_{i=1,2}(\Lambda_i \in (\widetilde{\Lambda}_{i-1}\ ,\ \widetilde{\Lambda}_i))$ such that $\{B_m(\Lambda_i)\}_{m \in \mathbb{Z}}$ satisfies

$$|B_0(\Lambda_i) - \mathfrak{B}(\Lambda_i)| = 1, \tag{4.8}$$

and there exists a constant $\Lambda_3 \in (\widetilde{\Lambda}_3\ ,\ \widetilde{\Lambda}_4)$ such that $\{D_m(\Lambda_3)\}_{m \in \mathbb{Z}}$ satisfies

$$D_0(\Lambda) = 2Re\mathfrak{D}(\Lambda). \tag{4.9}$$

And if $\Lambda \in (0\ ,\ \widetilde{\Lambda}_3]$, then it holds that $D_0(\Lambda) \neq 2Re\mathfrak{D}(\Lambda)$, and if $\Lambda \in [\widetilde{\Lambda}_2\ ,\ \widetilde{\Lambda}_5]$, then it holds that $|B_0(\Lambda) - \mathfrak{B}(\Lambda)| \neq 1$. Moreover, it holds that $\{b_m(\Lambda)\}_{m \in \mathbb{Z}}$ satisfying the equality (4.4) and $\{d_m(\Lambda)\}_{m \in \mathbb{Z}}$ satisfying the equality (4.5) are uniquely determined except for constant factor. Therefore $\Lambda_i^{-1}$ ( $i = 1, 2, 3$ ) are eigenvalues with simple multiplicity. This completes the proof of Proposition 3.2. $\square$

## 5. Numerical Computations

Results of the numerical computations agree well with our theorem. They are found in [3] with some graphics.

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